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Determinant of the Laplacian on a non-compact three-dimensional hyperbolic manifold with finite volume

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Abstract. The functional determinant of Laplace-type operators on a three-dimensional non-compact hyperbolic manifold with invariant fundamental domain of finite volume is expressed via the Selberg zeta function related to the Picard group $SL(2, \mathbb{Z} + i\mathbb{Z})/\{\pm \mathrm{Id}\}$.

1. Introduction

It is known that within the one-loop approximation, the Euclidean partition function in quantum field theory may be expressed as functional determinants associated with elliptic second order differential operators. Therefore, in recent years there have been many investigations into functional determinants on topologically non-trivial manifolds. Most of them have been concerned with Riemann flat or spherical spaces (see, for example, [1, 2] and references therein) or orbifold factors of spheres [3, 4]. The case of compact hyperbolic manifolds has also been considered (see for example [5, 6, 7, 8, 9, 10, 11, 2, 12, 13]). In this case one is dealing with two-dimensional H^2/Γ and three-dimensional H^3/Γ compact hyperbolic manifolds, H^N being the Lobachevsky space and Γ a discrete group of isometries acting on H^N and containing loxodromic, hyperbolic and elliptic elements (see [14, 15, 16, 17, 12]). Such manifolds are relevant in string theory and in cosmological scenarios.

For non-compact Riemannian surfaces of finite area, the functional determinant of Laplace operator has been computed in [18, 19]. Due to the potential relevance in $(1+2)$-dimensional quantum field theoretical models (Chern–Simons models), in this paper we extend the analysis to the three-dimensional case, considering a Laplace-type operator acting on functions in a non-compact, three-dimensional manifold H^3/Γ. In our example, the discrete group of isometry can be chosen in the form $SL(2, \mathbb{Z} + i\mathbb{Z})/\{\pm \mathrm{Id}\}$, a subgroup of the standard Picard group, Id being the identity element of Γ and it is associated with a non-compact manifold having an invariant fundamental domain of finite volume.

Making use of the Selberg trace formula, we shall investigate the asymptotic expansion of the heat kernel trace $\mathrm{Tr}\exp(-tL)$, L being a Laplace-like operator. Recall that one can

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define the spectral ζ-function $\zeta(s|L)$ associated with the operator L by means of the Mellin transform

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1}\, \mathrm{Tr}\, \mathrm{e}^{-tL}. \tag{1.1}$$

For $\mathrm{Re}\, s > \frac{3}{2}$ it is a well-defined analytic function and it can be continued analytically to a meromorphic function on the whole complex plane.

We shall find that the presence of parabolic elements in Γ leads to the appearance of a logarithmic factor in the small t asymptotic heat kernel expansion. This fact has been observed in [18, 19] for non-compact Riemannian surfaces of finite area. In this case the meromorphic continuation of the ζ-function has been shown to be regular at $s = 0$, thus the determinant of the Laplacian has been evaluated by means of the standard ζ-function regularization [20, 21]. In the three-dimensional case we shall show that $\zeta(s|L)$ is still a meromorphic function regular at $s = 0$, allowing the use of ζ-function regularization.

The contents of the paper are as follows. In section 2 we summarize some properties of the combined contributions to the Selberg trace formula needed in the paper. In section 3 the heat kernel trace and the ζ-function for a Laplace type operator are studied by making use of the trace formula. In section 4 the functional determinat is evaluated by means of the quadrature method. Finally we end with some conclusions in section 5.

2. Fundamental domain of the discrete group $SL(2, \mathbb{Z} + \mathrm{i}\mathbb{Z})/\{\pm \mathrm{Id}\}$ and the Selberg trace formula associated with the cusp form

Here we summarize the geometry and local isometry related to a simple three-dimensional complex Lie group. We shall consider discrete subgroup $\Gamma \subset SL(2, \mathbb{C})/\{\pm \mathrm{Id}\}$, where Id is the 2×2 identity matrix of the Γ. The group Γ acts discontinuously at the point $z \in \overline{\mathbb{C}}$, $\overline{\mathbb{C}}$ being the extended complex plane. We recall that a transformation $\gamma \neq \mathrm{Id}$, $\gamma \in \Gamma$, with

$$\gamma z = \frac{az+b}{cz+d} \qquad ad - bc = 1 \qquad (\mathrm{Tr}\,\gamma)^2 = (a+d)^2 \qquad a, b, c, d \in \mathbb{C} \tag{2.1}$$

is called elliptic if $(\mathrm{Tr}\,\gamma)^2$ satisfies $0 \leqslant (\mathrm{Tr}\,\gamma)^2 < 4$, hyperbolic if $(\mathrm{Tr}\,\gamma)^2 > 4$, parabolic if $(\mathrm{Tr}\,\gamma)^2 = 4$ and loxodromic if $(\mathrm{Tr}\,\gamma)^2 \in \mathbb{C}\backslash [0, 4]$. The classification of these transformations can also be based on the properties of their fixed points, the number of which is 1 for the parabolic transformations and 2 for all other cases.

The element $\gamma \in SL(2, \mathbb{C})$ acts on $p = (y, w) \in H^3$, $w = x_1 + \mathrm{i}x_2$ by means of the following linear-fractional transformation:

$$\gamma p = \left(\frac{y}{|cw+d|^2 + |c|^2 y^2}, \frac{(aw+b)(\bar{c}\,\bar{w} + \bar{d}) + a\bar{c}y^2}{|cw+d|^2 + |c|^2 y^2} \right). \tag{2.2}$$

The isometric circle of a transformation $\gamma \in SL(2, \mathbb{C})/\{\pm \mathrm{Id}\}$ for which ∞ is not a fixed point is defined to be the circle

$$I(\gamma) = \{z : \ |\gamma z| = 1\} \qquad \text{or} \qquad I(\gamma) = \{z : \ |z + d/c| = |c|^{-1}\} \quad c \neq 0. \tag{2.3}$$

A transformation γ carries its isometric circle $I(\gamma)$ into $I(\gamma^{-1})$.

The isometric fundamental domain of a Fuchsian group (Kleinian group without loxodromic elements) has the following structure: it is bounded by arcs of circles orthogonal to the invariant circle and consists either of two symmetric components or of a single component, while the mappings connecting its equivalent sides generate the whole group. In many cases, it is more convenient to deal with other fundamental regions. For example, the so-called normal fundamental Dirichlet polygons are often used for Fuchsian groups and we shall follow this approach here.

Now we consider a discrete subgroup of a special kind. Let $G = PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\{\pm\,\mathrm{Id}\}$, then for $\Gamma \subset G$, one can choose Γ in the form $SL(2,\mathbb{Z}+\mathrm{i}\mathbb{Z})/\{\pm\,\mathrm{Id}\}$, where $\mathbb{Z}$ is the ring of integer numbers. The group Γ has, within a conjugation, one maximal parabolic subgroup Γ_∞ ($c = 0$). Thus, the fundamental domain related to Γ has one parabolic vertex and can be taken in the form [22, 23]

$$F(\Gamma) = \left\{(y,w) : x_1^2 + x_2^2 + y^2 > 1, \quad -\tfrac{1}{2} < x_2 < x_1 < \tfrac{1}{2}\right\}. \tag{2.4}$$

Remark. Let a free Abelian group of isometries be generated by the two parabolic mappings

$$g_1(z) = z + 1 \qquad g_2(z) = z + \mathrm{i} \tag{2.5}$$

then, if we identify the faces of the polyhedron, equation (2.4), we obtain a manifold $M(\Gamma)$ that is homeomorphic to a punctured torus $S^1 \otimes S^1 \otimes [-\frac{1}{2}, \frac{1}{2}) = U_c \otimes S^1$, where $U_c = \{z : \ 0 < |z| \leqslant \frac{1}{2}\}$ is a punctured cylinder. It is turned into a hyperbolic manifold by removing the boundary $\partial M(\Gamma)$, which is homeomorphic to the torus $S^1 \otimes S^1$.

Now we are ready to start discussing the Selberg trace formula, which can be constructed as an expansion in eigenfunctions of the automorphic Laplacian. To begin with, we assume that the group Γ has a cusp at ∞ ($c = 0$), each element of the stabilizer Γ_∞ is a translation. Computing the conjugacy class $\{\gamma\}_\Gamma$, $\gamma \in \Gamma_\infty$ with γ different from the identity, one easily obtains the following proposition.

Proposition 1. Let

$$\gamma = \begin{pmatrix} 1 & n_1 + \mathrm{i}n_2 \\ 0 & 1 \end{pmatrix} \qquad n_1, n_2 \in \mathbb{Z}. \tag{2.6}$$

The conjugacy class with representative γ consists in element γ and γ^{-1}, where

$$\gamma^{-1} = \begin{pmatrix} 1 & -n_1 - \mathrm{i}n_2 \\ 0 & 1 \end{pmatrix}. \tag{2.7}$$

The remaining conjugacy classes have the representatives in Γ_∞ of the form

$$\gamma_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} \mathrm{i} & 1 \\ 0 & -\mathrm{i} \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 0 & -\mathrm{i} \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} \mathrm{i} & 1-\mathrm{i} \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{2.8}$$

The centralizers related to these representations read

$$\begin{aligned}
\Gamma^\gamma &= \begin{pmatrix} 1 & m_1 + \mathrm{i}m_2 \\ 0 & 1 \end{pmatrix} \qquad m_1, m_2 \in \mathbb{Z} \\
\Gamma^1 = \Gamma^{\gamma_1} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ \mathrm{i} & 0 \end{pmatrix} \right\} \\
\Gamma^2 = \Gamma^{\gamma_2} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 1 \\ 0 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 2 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} -1 & \mathrm{i} \\ 2\mathrm{i} & 1 \end{pmatrix} \right\} \\
\Gamma^3 = \Gamma^{\gamma_3} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 0 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 2\mathrm{i} & -\mathrm{i} \end{pmatrix} \right\} \\
\Gamma^4 = \Gamma^{\gamma_4} &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 1-\mathrm{i} \\ 0 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 1+\mathrm{i} & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 & -1-\mathrm{i} \\ 1-\mathrm{i} & -1 \end{pmatrix} \right\}.
\end{aligned} \tag{2.9}$$

Let us consider an arbitrary integral operator with kernel $k(z, z')$. Invariance of the operator is equivalent to fulfillment of the condition $k(\gamma z, \gamma z') = k(z, z')$ for any $z, z' \in H^3$ and $\gamma \in PSL(2, \mathbb{C})$. So the kernel of the invariant operator is a function of the geodesic distance between z and z'. It is convenient to replace such a distance with the fundamental invariant of a pair of points $u(z, z') = |z - z'|^2/yy'$, thus $k(z, z') = k(u(z, z'))$. Let λ_j be the isolated eigenvalues of the self-adjoint extension of the Laplace operator and let us introduce a suitable analytic function $h(r)$ and $r_j^2 = \lambda_j - 1$. It can be shown that $h(r)$ is related to the quantity $k(u(z, \gamma z))$ by means of the Selberg transform. Let us denote by $g(u)$ the Fourier transform of $h(r)$, namely

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ru} h(r)\, \mathrm{d}r. \tag{2.10}$$

For one parabolic vertex let us introduce a subdomain F_Y of the fundamental region $F(\Gamma)$ by means

$$F_Y = \{z \in F(\Gamma), z = \{y, \boldsymbol{x}\} \,|\, y \leqslant Y\} \tag{2.11}$$

where Y is a sufficiently large positive number.

Lemma 1. Suppose $h(r)$ to be an even analytic function in the strip $|\operatorname{Im} r| < 1 + \varepsilon$ $(\varepsilon > 0)$ and $h(r) = \mathrm{O}(1 + |r|^2)^{-2}$. Then for $N = 3$ the following formula holds [22]:

$$\sum_j h(r_j) = \lim_{Y \to \infty} \left\{ \int_{F_Y} \sum_{\{\gamma\}_\Gamma} k(u(z, \gamma z))\, \mathrm{d}\mu(z) - \frac{1}{2\pi} \int_0^\infty h(r) \int_{F_Y} |E(z, 1 + \mathrm{i}r)|^2\, \mathrm{d}\mu(z)\, \mathrm{d}r \right\} \tag{2.12}$$

where $\mathrm{d}\mu(z) = y^{-3}\mathrm{d}y\mathrm{d}x_1\mathrm{d}x_2$ is the invariant measure on H^3 and $E(z, s)$ is the Eisenstein–Maass series associated with one cusp, namely

$$E(z, s) = \sum_{\gamma \in (\Gamma/\Gamma_\infty)} y^s(\gamma z) \qquad x_2(z) = \operatorname{Im} z. \tag{2.13}$$

The series (2.13) converges absolutely for $\operatorname{Re} s > 1$ and uniformly in z on compact subset of H^3. All poles of $E(z, s)$ are contained in the union of the region $\operatorname{Re} s < \frac{1}{2}$ and the interval $\left[\frac{1}{2}, 1\right]$ and those contained in such an interval are simple. Furthermore, for each s, the series $E(z, s)$ is a real analytic function on H^3, automorphic relative to the group Γ and satisfies the eigenvalues equation

$$\Delta E(z, s) = s(s - 1)E(z, s) \tag{2.14}$$

Δ being the Laplace operator. The asymptotic expansion of the second integral in (2.12) can be found with the help of Maass–Selberg relation [22]. For $Y \to \infty$ one has

$$\begin{aligned} &\frac{1}{2\pi} \int_0^\infty h(r) \int_{F_Y} |E(z, 1 + ir)|^2\, \mathrm{d}\mu(z)\, \mathrm{d}r \\ &\quad = g(0) \ln Y + \frac{h(0)}{4} S(1) - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{S'(1 + \mathrm{i}r)}{S(1 + \mathrm{i}r)}\, \mathrm{d}r + \mathrm{O}(1). \end{aligned} \tag{2.15}$$

The function $S(s)$ (in the general case it is the S-matrix) is given by a generalized Dirichlet series, convergent for $\operatorname{Re} s > 1$,

$$S(s) = \frac{\sqrt{\pi}\,\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c \neq 0} \sum_{0 \leqslant d < |c|} |c|^{-2s} \tag{2.16}$$

where the sums are taken over all pairs c, d of the matrix $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \subset \Gamma_\infty \backslash \Gamma / \Gamma_\infty$. Also, the poles of the meromorphic function $S(s)$ are contained in the region $\operatorname{Re} s < \frac{1}{2}$ and in the interval $\left[\frac{1}{2}, 1\right]$. The functions $E(z, s)$ and $S(s)$ can be extented analytically on the whole complex s-plane, where they satisfy the functional equations

$$S(s)S(1-s) = \mathrm{Id} \qquad \overline{S(s)} = S(\bar{s}) \qquad E(z, s) = S(s)E(z, 1-s). \tag{2.17}$$

It should be noted that the terms of the trace formula associated with the elements γ and γ^{-1} coincide. Then the contribution to the first integral in (2.12), which comes from all γ type conjugacy classes ($\gamma \in \Gamma^{\gamma}$), given in proposition 1, for $Y \to \infty$ can be written as follows:

$$\int_{F_Y} \sum_{\{\gamma\}_{\Gamma_\infty}} k(u(z, \gamma z))\, \mathrm{d}\mu(z)$$

$$= (\ln Y + C)\, g(0) + \frac{h(0)}{4} - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\psi(1 + \tfrac{1}{2}\mathrm{i}r)\, \mathrm{d}r + \mathrm{O}(1) \tag{2.18}$$

where $\psi(s)$ is the logarithmic derivative of the Euler Γ-function and C is a computable constant which reads

$$C = \tfrac{5}{16} \ln 2 - \tfrac{1}{2}\gamma + C_0$$

$$C_0 = \lim_{N\to\infty} \frac{1}{4\pi} \sum_{m=1}^{N} \left[|\xi^{(m)}|^{-2} - 2\pi \ln \frac{|\xi^{(m+1)}|}{|\xi^{(m)}|} \right]. \tag{2.19}$$

In the latter equation γ is the Euler–Mascheroni constant and $\xi^{(m)}$ is a sequence of two-dimensional vectors such that $|n_1| + |n_2| = m$, $\gamma z = \{y, \omega + \xi\}$, $\xi \neq 0$, $|\xi^{(m+1)}| \geqslant |\xi^{(m)}|$ [22].

For the derivation of the Selberg trace formula, one has to consider the contributions coming from the identity and the non-parabolic elements in Γ, the normalized Eisenstein–Maass series, equation (2.13), and all γ-type conjugacy classes, equation (2.18). The final result we state should be considered as an explicit addition to lemma 1 [22].

Theorem 1. For the special discrete group $SL(2, \mathbb{Z} + \mathrm{i}\mathbb{Z})/\{\pm \mathrm{Id}\}$ and $h(r)$ satisfying the conditions of lemma 1, we have the Selberg trace formula

$$\sum_j h(r_j) - \sum_{\substack{\{\gamma\}_\Gamma, \gamma\neq \mathrm{Id}, \\ \gamma=\text{non-parabolic}}} \int k(u(z, \gamma z))\, \mathrm{d}\mu(z) - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{S'(1+\mathrm{i}r)}{S(1+\mathrm{i}r)}\, \mathrm{d}r + \frac{h(0)}{4} S(1)$$

$$= V(F) \int_0^{\infty} \frac{r^2}{2\pi^2} h(r)\, \mathrm{d}r + Cg(0) + \frac{h(0)}{4} - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)\psi(1 + \tfrac{1}{2}\mathrm{i}r)\, \mathrm{d}r. \tag{2.20}$$

The first term on the RHS of (2.20) is the contribution of the identity element, while $V(F)$ is the (finite) volume of the fundamental domain F with respect to the measure $\mathrm{d}\mu$.

3. The heat kernel and the ζ-function

As discussed in the introduction, the determinant of an elliptic differential operator requires a regularization. It is convenient to introduce the operator $L_\delta = -\Delta + \delta^2 - 1$, with δ such that $\delta^2 > 1$. One of the most used regularization is the ζ-function regularization [20, 24, 21]. By this one has

$$\ln \det L_\delta = -\zeta'(0|L_\delta) \tag{3.1}$$

where ζ' is the derivative with respect to s of the ζ-function. In the standard cases, the ζ-function at $s = 0$ is well defined, and so by means of the latter formula one obtains a finite result.

We recall that the meromorphic structure of the analytically continued ζ-function, as well as the ultraviolet divergences of the one-loop effective action, can be related to the asymptotic properties of the heat-kernel trace. For the rank-1 symmetric space H^3/Γ the trace of the operator $\exp\left[-(tL_\delta)\right]$ may be computed by using theorem 1 (equation (2.20)) with the choice $h(r) = \exp\left[-t(r^2+\delta^2)\right]$ (we use units in which the curvature $\kappa = R/6$ of H^3 is equal to -1). We have

$$g(u) = \frac{\mathrm{e}^{-t\delta^2}\mathrm{e}^{-u^2/4t}}{\sqrt{4\pi t}} \qquad g(0) = \frac{\mathrm{e}^{-t\delta^2}}{\sqrt{4\pi t}} \qquad h(0) = \mathrm{e}^{-t\delta^2}. \tag{3.2}$$

In this and the following sections we shall consider additive terms of the ζ-function associated with identity and parabolic elements of the group Γ only (the heat kernel and ζ-function analysis for co-compact discrete group Γ has been carried out in [2, 12], for example).

As a result

$$\mathrm{Tr}\,\mathrm{e}^{-tL_\delta} = \mathrm{e}^{-t\delta^2}\left[\frac{V(F)}{(4\pi t)^{3/2}} + \frac{C}{(4\pi t)^{1/2}} + \frac{1}{4} - \frac{1}{4\pi}\int_{-\infty}^{\infty}\psi(1+\tfrac{1}{2}\mathrm{i}r)\mathrm{e}^{-tr^2}\,\mathrm{d}r\right]. \tag{3.3}$$

The asymptotic behaviour of the last integral for $t \to 0$ can be easily evaluated. In fact, by making an integration by parts, using (see [25])

$$\ln\Gamma(z) = z\ln z - z - \ln\sqrt{\frac{z}{2\pi}} + f(z)w \tag{3.4}$$

with

$$f(z) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{k}{(k+1)(k+2)}\sum_{n=1}^{\infty}\frac{1}{(n+z)^{k+1}} \tag{3.5}$$

and performing elementary integration we have

$$\mathrm{Tr}\,\mathrm{e}^{-tL_\delta} = \mathrm{e}^{-t\delta^2}\left[\frac{\ln t}{8\sqrt{\pi t}} + \frac{V(F)}{(4\pi t)^{3/2}} + \frac{C+\ln 2+\frac{1}{4}\gamma}{(4\pi t)^{1/2}} - \frac{t}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{e}^{-tr^2}f(\tfrac{1}{2}\mathrm{i}r)\,\mathrm{d}r\right]. \tag{3.6}$$

The function $f(z)$ has an aymptotic expansion for large $|z|$ in terms of the Bernoulli numbers B_k given by [25]

$$f(z) \sim \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)z^{2k-1}}. \tag{3.7}$$

The contribution for short t comes from these asymptotics. Thus we have

Proposition 2. The asymptotic behaviour of the heat kernel for $t \to 0$ reads

$$\mathrm{Tr}\,\mathrm{e}^{-tL_\delta} \simeq \mathrm{e}^{-t\delta^2}\left[\frac{\ln t}{8\sqrt{\pi t}} + \sum_{n=0}^{\infty}K_n t^{n-3/2}\right] = \sum_{r=0}^{\infty}(A_r + P_r\ln t)t^{r-3/2} \tag{3.8}$$

where the first K_n coefficients are given by

$$K_0 = \frac{V(F)}{(4\pi)^{3/2}} \qquad K_1 = \frac{C+\ln 2+\frac{1}{4}\gamma}{\sqrt{4\pi}} \qquad K_2 = \frac{1}{6\sqrt{\pi}} \tag{3.9}$$

and

$$A_r = \sum_{n=0}^{r}(-1)^n\frac{B_{r-n}\delta^{2n}}{n!} \qquad P_0 = 0 \qquad P_r = (-1)^{r-1}\frac{\delta^{2(r-1)}}{8\sqrt{\pi}(r-1)!}. \tag{3.10}$$

It should be noted that, besides the usual terms one has for the heat kernel in three-dimensions, there exist terms with logarithmic factors due to the presence of parabolic elements in Γ. These terms are absent for the co-compact group Γ (compact hyperbolic manifolds). Furthermore, the contribution of the hyperbolic elements is exponentially small in t. Thus, the result of proposition 2 still holds true.

Let us analyse the consequences of the presence of logarithmic terms in the latter expansion. We recall that the ζ-function associated with the elliptic operator L_δ is given by

$$\zeta(s|L_\delta) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\; t^{s-1}\, \mathrm{Tr}\, \mathrm{e}^{-tL_\delta} \tag{3.11}$$

valid for $\mathrm{Re}\, s > \frac{3}{2}$. In order to obtain the meromorphic structure of the function (3.11), we split the integration range in the two intervals $[0, 1)$ and $[1, \infty)$, in this way obtaining two integrals. The latter is regular for $s \to 0$, while the behaviour of the former one can be estimated by using the asymptotics, equation (3.8). Thus we have

Proposition 3. The meromorphic structure of the ζ-function reads

$$\zeta(s|L_\delta) = \frac{1}{\Gamma(s)} \sum_{r=0}^{\infty} \left[\frac{A_r}{s + r - \frac{3}{2}} - \frac{P_r}{(s + r - \frac{3}{2})^2} \right] + \frac{J(s)}{\Gamma(s)} \tag{3.12}$$

where $J(s)$ is an analytic function.

From this, it follows that the analytic continuation of ζ-function is regular at $s = 0$. The presence of double poles, caused by the logarithmic terms, must also be noted.

We conclude this section by computing the asymptotic behaviour for very large δ of the derivative of the ζ-function evaluated at zero. To this end, the asymptotic behaviour for small t [7] again gives

Proposition 4.

$$\zeta'(0|L_\delta) = \frac{V(F)\delta^3}{6\pi} + \frac{1}{2}\delta \ln \delta - \delta \left(C + \frac{1}{2} \ln 2 + \frac{1}{2} \right) + \mathrm{O}\left(\frac{1}{\delta}\right). \tag{3.13}$$

4. The functional determinant

In this section, making use of the trace formula, we shall compute the functional determinant of a Laplace-type operator on H^3/Γ. We briefly explain the method which is based on ζ-function regularization and an evaluation by quadratures with an appropriate choice of the function $h(r)$ appearing in the trace formula [6, 7, 18]. The ζ-function, for $\mathrm{Re}\, s > \frac{3}{2}$, can be rewritten in the form

$$\zeta(s|L_\delta) = \sum_\sigma \rho_\sigma \left(\lambda_\sigma + \delta^2 - 1\right)^{-s} = \sum_j \left(\lambda_j + \delta^2 - 1\right)^{-s} + \int_0^\infty \left(\lambda + \delta^2 - 1\right)^{-s} \rho_\lambda \,\mathrm{d}\lambda \tag{4.1}$$

where the sum over j runs over the discrete spectrum, λ_j being the eingenvalues and we put $\rho_i = 1$ here. For the continuous spectrum, ρ_λ is proportional to the logarithmic derivative of the S-matrix $S(s)$. One has

$$\zeta'(s|L_\delta) = -\sum_\sigma \rho_\sigma \left(\lambda_\sigma + \delta^2 - 1\right)^{-s} \ln(\lambda_\sigma + \delta^2 - 1). \tag{4.2}$$

From the latter equation one obtains

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{1}{2\delta}\frac{\mathrm{d}}{\mathrm{d}\delta}\zeta'(s|L_\delta)\right) = 2\delta \sum_\sigma \rho_\sigma \left(\lambda_\sigma + \delta^2 - 1\right)^{-s-2} + \mathrm{O}(s). \tag{4.3}$$

A standard Tauberian argument and equation (3.8) gives a Weyl estimate for large σ, namely $(\lambda_\sigma + \delta^2 - 1)^{-1} \simeq \sigma^{-2/3}$. As a consequence, in the limit $s \to 0$, the RHS of (4.3) is finite. This works for dimensions $D = 2$ as well as for $D = 3$. In higher dimensions it is necessary to take further derivatives with respect to δ [7].

On the other hand, we may rewrite equation (2.20) (here $r^2 + 1 = \lambda$ and γ is the identity or a parabolic element in Γ) as

$$G(\delta) = \sum_\sigma \rho_\sigma h_\delta(r_\sigma) = \sum_j h_\delta(\lambda_j) - \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\delta(r)\frac{S'(1+\mathrm{i}r)}{S(1+\mathrm{i}r)}\,\mathrm{d}r + \frac{h_\delta(0)}{4}S(1). \tag{4.4}$$

$G(\delta)$ denoting the 'geometrical' part

$$G(\delta) = V(F)\int_0^{\infty} \frac{r^2}{2\pi^2}\,h_\delta(r)\,\mathrm{d}r + Cg(0) + \frac{h_\delta(0)}{4} - \frac{1}{4\pi}\int_{-\infty}^{\infty} h_\delta(r)\psi(1+\tfrac{1}{2}\mathrm{i}r)\,\mathrm{d}r. \tag{4.5}$$

Let us choose the function h_δ as

$$h_\delta(r) = \frac{1}{r^2+\delta^2} - \frac{1}{r^2+a^2} \qquad g_\delta(0) = \frac{1}{2\delta} - \frac{1}{2a} \tag{4.6}$$

with a a non-vanishing constant. Taking the derivative with respect to δ we have

$$2\delta\sum_\sigma \rho_\sigma\left(\lambda_\sigma + \delta^2 - 1\right)^{-2} = -\frac{\mathrm{d}}{\mathrm{d}\delta}G(\delta). \tag{4.7}$$

Making the comparison between equation (4.3) and (4.7), integrating twice in the variable δ and taking the limit $s \to 0$, we finally obtain

$$\zeta'(0|L_\delta) = -2\int \delta G(\delta)\,\mathrm{d}\delta + c_1\delta^2 + c_2 \tag{4.8}$$

where the constants c_1 and c_2 can be determined from the asymptotics for large δ. The primitive related to the geometrical part can be easily computed by making use of the Selberg trace formula with the choice (4.6). One has

$$G(\delta) = -\frac{V(F)}{4\pi}(\delta - a) + \frac{C}{2}\left(\frac{1}{\delta} - \frac{1}{a}\right) + \frac{1}{4}\left(\frac{1}{\delta^2} - \frac{1}{a^2}\right)$$
$$- \frac{1}{4}\left[\frac{\psi(1+\delta/2)}{\delta} - \frac{\psi(1+a/2)}{a}\right]. \tag{4.9}$$

As a consequence

$$\zeta'(0|L_\delta) = \frac{V(F)\delta^3}{6\pi} + [c_1 + Q(a)]\delta^2 - C\delta - \frac{1}{2}\ln\delta + \ln\Gamma(1+\tfrac{1}{2}\delta) + c_2 \tag{4.10}$$

where

$$Q(a) = \frac{1}{4a^2} - \frac{V(F)a}{4\pi} + \frac{C}{2a} - \frac{\psi(1+a/2)}{4a}. \tag{4.11}$$

The inclusion of the contribution related to the hyperbolic elements in (4.10) is almost straightforward and can be found in [2, 12]. It is additive and reads simply $\ln Z(1+\delta)$, $Z(s)$ being the Selberg zeta-function, which may defined by means of its logarithmic derivative; namely for $\mathrm{Re}\, s > 1$ one has [16, 12]

$$\frac{Z'(s)}{Z(s)} = \sum_{\{P\}}\sum_{k=1}^{\infty} \frac{\chi(P(\gamma)^k)}{S_3(k; l_\gamma)}\exp\left(-(s-1)kl_\gamma\right) \tag{4.12}$$

where $\{P\}$ is a set of primitive closed geodesics, $P(\gamma)$ being the holonomy element by parallel traslation around the $\gamma \in \Gamma$ element of the conjugacy class, l_γ the corresponding geodesic length and $S_3(k; l_\gamma)$ a known function of the conjugacy class. In the large δ limit, $\ln Z(1+\delta)$ is vanishing and one has

$$\zeta'(0|L_\delta) \simeq \frac{V(F)\delta^3}{6\pi} + [c_1 + Q(a)]\delta^2 + \frac{1}{2}\delta\ln\delta - \delta\left(C + \frac{1}{2}\ln 2 + \frac{1}{2}\right) + \frac{1}{2}\ln\pi + c_2 \tag{4.13}$$

which agrees with (3.13) if

$$c_1 = -Q(a) \qquad c_2 = -\tfrac{1}{2}\ln\pi. \tag{4.14}$$

Summarizing we have proved the following theorem.

Theorem 2. One has the identity

$$\det L_\delta = \frac{\sqrt{\pi\delta}}{\Gamma(1+\frac{1}{2}\delta)} \exp\left(-\frac{V(F)\delta^3}{6\pi} + C\delta\right) Z(1+\delta). \tag{4.15}$$

5. Conclusions

In this paper we have computed the functional determinant of a Laplace-like operator on a non-compact three-dimensional hyperbolic manifold with finite volume fundamental domain by the method of quadratures. In addition, the contributions to the heat kernel and the ζ-function associated with the identity and parabolic elements of the isometry group have been analysed. The constant appearing in the quadrature process has been determined by means of the asymptotic behaviour of the functional determinant, which may be achived again making use of the trace formula for the heat kernel. This method is particular useful in the evaluation of the functional determinants, because it allows one to avoid the problem of finding the analytical continuation of the zeta-function, which may present computational difficulties. On the other hand, the method requires the existence of a trace formula and its validity can be extended to more general cases (see, for example, [26, 27, 28]).

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